Well-posedness and Regularity for Distribution Dependent SPDEs with Singular Drifts

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Outline

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Introduction

Consider the following SDE (called stochastic McKean-Vlasov equation/ mean field equation/ distribution dependent equation):

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t,$$

where $\mathscr{L}_{X_t}$ denotes the law of X_t.

Introduction

Some Known Results for DDS(P)DEs:

- Existence and Uniqueness of Solutions: Funaki(ZWVG,'84), Gradham(SPA,'92), Dawson,Vaillancourt(NDEA,'95), Kotelenez, Kurtz(PTRF,'10), Huang, Wang(SPA,'19), Röckner, Zhang(Bernoulli,'20), Li, Li, Xie(JSP,'20)
- Nonlinear F-P: Huang, Röckner, Wang(DCDS,'19), Barbu, Röckner (SIAM-JMA,'18;AOP,'20), Röckner,Xie,Zhang(PTRF,'20)
- Regularity: Wang(SPA,'18), Crisan, McMurray(PTRF,'18), Baños(AIHP,'18), S.(JTP,'20,CPAA,'21+), Röckner, Zhang(Bernoulli '20), Ren, Wang(JDE,'19), Huang, Wang(SPA,'19)
- Functional inequalities: Guillin, Liu, Wu, Zhang(AAP,'20+), Ren, Wang (NLA,'20)
- Ergodicity, Propagation of chaos: Guillin, Liu, Wu, Zhang(AAP,'20+), Ren, Wang (NLA,'20).
- J. Bao, C. Deng, X. Fan, W. Liu, J. Shao, J. Wang, S. Zhang and so on.

Distribution Dependent SPDEs

Let $(\mathbb{H}, \langle , \rangle, |\cdot|)$ and $(\bar{\mathbb{H}}, \langle , \rangle_{\bar{\mathbb{H}}}, |\cdot|_{\bar{\mathbb{H}}})$ be two separable Hilbert spaces, and $\{W_t\}_{t\geq 0}$ be a cylindrical Brownian motion on $\bar{\mathbb{H}}$ with respect to a complete filteration probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$.

Let $\mathcal{P}$ be the set of all probability measures on $\mathbb{H}$ equipped with the weak topology. Consider the following semi-linear distribution dependent SDEs on $\mathbb{H}$:

$$\mathrm{d}X_t = \{AX_t + b_t(X_t, \mathscr{L}_{X_t})\}\mathrm{d}t + Q_t(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t, \tag{1}$$

where $(A, \mathcal{D}(A))$ is a negative definite self-adjoint operator on $\mathbb{H}$, $b : \mathbb{R}_+ \times \mathbb{H} \times \mathcal{P} \to \mathbb{H}$ and $Q : \mathbb{R}_+ \times \mathbb{H} \times \mathcal{P} \to \mathcal{L}(\bar{\mathbb{H}}; \mathbb{H})$ are measurable.

Aims of this talk:

♣ The existence and uniqueness of strong and weak solutions

♣ Wang's Log-Harnack inequality, Harnack inequality and shift Harnack inequality

Outline

Notations

Define

$$\mathcal{P}_2 := \left\{ \mu \in \mathcal{P} : \mu(|\cdot|^2) := \int_{\mathbb{H}} |x|^2 \mu(\mathrm{d}x) < \infty \right\},$$

which is a Polish space under the Wasserstein distance

$$\mathbb{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathfrak{C}(\mu_1, \mu_2)} \left(\int_{\mathbb{H}\times\mathbb{H}} |x-y|^2 \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{2}},$$

where $\mathfrak{C}(\mu_1, \mu_2)$ is the set of all couplings for μ_1 and μ_2.

Definition of Solutions

Definition 1

A continuous $\{\mathscr{F}_t\}$-adapted process $\{X_t\}_{t\geq 0}$ is called a mild solution, if $\mathbb{P}$-a.s

$$X_t = e^{At}X_0 + \int_0^t e^{A(t-s)} b_s(X_s, \mathscr{L}_{X_s})\mathrm{d}s + \int_0^t e^{A(t-s)} Q_s(X_s, \mathscr{L}_{X_s})\mathrm{d}W_s, \quad t \geq 0. \quad (2)$$

Moreover, if $\mathbb{E}|X_t|^2 < \infty$ for any $t \geq 0$, then the solution is said in $\mathcal{P}_2$. Equ. (1) is called strongly well-posed in $\mathcal{P}_2$, if for any $\mathscr{F}_0$-measurable random variable X_0 with $\mathcal{L}_{X_0} \in \mathcal{P}_2$, there exists a unique mild solution in $\mathcal{P}_2$.

Definition of Solutions

Definition 2

- A couple $(\tilde{X}_t, \tilde{W}_t)_{t\geq 0}$ is called a weak solution to Equ. (1), if $\tilde{W}$ is a cylindrical Brownian motion with respect to a complete filtered probability space $(\tilde{\Omega}, \tilde{\mathscr{F}}, \tilde{\mathbb{P}}; \{\tilde{\mathscr{F}}_t\}_{t\geq 0})$, and (2) holds for $(\tilde{X}_t, \tilde{W}_t)_{t\geq 0}$ in place of $(X_t, W_t)_{t\geq 0}$. Moreover, if $\mathscr{L}_{\tilde{X}_t}|_{\tilde{\mathbb{P}}} \in \mathcal{P}_2$, the weak solution is called in $\mathcal{P}_2$.

- Equ.(1) is said to have weak uniqueness in $\mathcal{P}_2$, if any two weak solutions in $\mathcal{P}_2$ of (1) from common initial distribution are equal in law. Furthermore, we call weak well-posedness in $\mathcal{P}_2$ for Equ.(1) holds, if it has a weak solution from any initial distribution and has weak uniqueness in $\mathcal{P}_2$.

Notations

Denote

$$\mathscr{D} = \Big\{\phi : \mathbb{R}_+ \to \mathbb{R}_+ | \phi^2 \text{ is concave and } \phi \text{ is increasing with } \int_0^1 s^{-1}\phi(s)\mathrm{d}s < \infty$$

Assumptions

There exists an increasing function $K : (0, \infty) \to (0, \infty)$ such that A, b and Q satisfy the following conditions.

(a1) For some $\varepsilon \in (0, 1)$, $(-A)^{\varepsilon - 1}$ is of trace class.

(a2) The operator $Q : [0, \infty) \times \mathbb{H} \times \mathcal{P} \to \mathcal{L}(\bar{\mathbb{H}}; \mathbb{H}))$ is continuous and for each $t \geq 0$ and $\mu \in \mathcal{P}$, and $Q_t(\cdot, \mu)$ is in $C^2(\mathbb{H}; \mathcal{L}(\bar{\mathbb{H}}; \mathbb{H}))$ such that

$$\sup_{(t,x,\mu) \in [0,T] \times \mathbb{H} \times \mathcal{P}_2} \Big(\|Q_t(x, \mu)\| + \|\nabla Q_t(x, \mu)\| + \|\nabla^2 Q_t(x, \mu)\| \Big) \leq K(T),$$

Meanwhile, $(Q_t Q_t^*)(x, \mu)$ is invertible for each $(t, x, \mu) \in [0, \infty) \times \mathbb{H} \times \mathcal{P}_2$ with

$$\sup_{(t,x,\mu) \in [0,T] \times \mathbb{H} \times \mathcal{P}} \|(Q_t Q_t^*)(x, \mu)^{-1}\| \leq K(T).$$

Moreover, for any $x \in \mathbb{H}$, $t \geq 0$ and $\mu \in \mathcal{P}_2$, it holds

$$\lim_{n\to\infty} \|Q_t(x,\mu) - Q_t(\pi_n x,\mu)\|_{\mathrm{HS}}^2 = 0.$$

In addition, for any $T > 0$, it holds

$$\sup_{(t,x)\in[0,T]\times\mathbb{H}} \|Q_t(x,\mu) - Q_t(x,\nu)\|_{\mathrm{HS}}^2 \leq K(T)\mathbb{W}_2(\mu,\nu)^2, \quad \mu,\nu \in \mathcal{P}_2.$$

(a3) For any $t \in [0, T]$, b_t is continuous in $\mathbb{H} \times \mathcal{P}$. The function $t \mapsto \sup_{(x,\mu)\in\mathbb{H}\times\mathcal{P}_2} |b_t(x,\mu)|$ is locally bounded, and there exists $\phi \in \mathscr{D}$ such that

$$|b_t(x,\mu) - b_t(y,\nu)| \leq \phi(|x-y|) + K(t)\mathbb{W}_2(\mu,\nu), \quad t \geq 0, x, y \in \mathbb{H}, \mu,\nu \in \mathcal{P}_2.$$

Weak Solutions

Theorem 1[X.Huang, S., NLA, '21]

Assume **(a1)**. If $\sup_{(x,\mu)\in\mathbb{H}\times\mathcal{P}}(|b_t(x,\mu)|+\|Q_t(x,\mu)\|)$ is locally bounded with respect to t and b_t, Q_t are continuous in $\mathbb{H}\times\mathcal{P}$ for each $t\geq 0$, then for any fixed $T>0$, and $\mu_0\in\mathcal{P}$, Equ. (1) has a weak solution up to time T with initial distribution μ_0.

Weak Solutions

Outline of Proof:

Step 1. For each $n \geq 1$, let $\eta_n(s) = \lfloor \frac{s}{T/n} \rfloor \frac{T}{n}$, where $\lfloor \cdot \rfloor$ stands for the integer part. Let X_0 be an $\mathscr{F}_0$-measurable random variable with $\mathscr{L}_{X_0} = \mu_0$. For $t \in [0, T]$, define

$$X_t^n = e^{At} X_0 + \int_0^t e^{A(t-s)} b_s(X^n_{\eta_n(s)}, \mathscr{L}_{X^n_{\eta_n(s)}}) \mathrm{d}s + \int_0^t e^{A(t-s)} Q_s(X^n_{\eta_n(s)}, \mathscr{L}_{X^n_{\eta_n(s)}}) \mathrm{d}W_s.$$

Step 2. Prove $\{\mathscr{L}_{X^n}\}_{n \geq 1}$ is tight in the space of probability measures on $C([0, T]; \mathbb{H})$.

Step 3. By the Skorohod representation theorem and the martingale representation theorem, there exists a complete filtered probability space $(\hat{\Omega}, \hat{\mathscr{F}}, \{\hat{\mathscr{F}}_t\}, \hat{\mathbb{P}})$, a cylindrical Brownian motion $\tilde{\tilde{W}}$ and a continuous process $\tilde{X}$ such that

$$\tilde{X}_t = e^{At} \tilde{X}_0 + \int_0^t e^{A(t-s)} b_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s}|_{\hat{\mathbb{P}}}) \mathrm{d}s + \int_0^t e^{A(t-s)} Q_s(\tilde{X}_s, \mathscr{L}_{\tilde{X}_s}|_{\hat{\mathbb{P}}}) \mathrm{d}\tilde{\tilde{W}}_s, \quad t \in [0, T].$$

Existence and Uniqueness of Solutions

Theorem 2 [X.Huang, S., NLA, '21]

Assume **(a1)-(a3)**. Then the following assertions hold.

(1) Equ.(1) has weak well-posedness in $\mathcal{P}_2$ and there exists a constant $C(T) > 0$ such that

$$\int_0^T \mathbb{W}_2(P_t^*\mu_0, P_t^*\nu_0)^2 \mathrm{d}t \le C(T)\mathbb{W}_2(\mu_0,\nu_0)^2, \quad \mu_0,\nu_0 \in \mathcal{P}_2.$$

(2) The strong well-posedness in $\mathcal{P}_2$ holds for Equ.(1). Moreover, there exists an increasing function $C : [0,\infty) \to [0,\infty)$ such that for any two solutions X_t and Y_t to Equ.(1), it holds

$$\int_0^T \mathbb{E}|X_t - Y_t|^2 \mathrm{d}t \le C(T)\mathbb{E}|X_0 - Y_0|^2, \quad T \ge 0.$$

Modified Yamada-Watanabe Principle

Consider DDSDEs on $\mathbb{R}^d$,

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t \tag{3}$$

Lemma[1]

Assume that (3) has a weak solution $\{\overline{X}_t\}_{t\in[0,T]}$ under probability $\overline{\mathbb{P}}$. If the SDE

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{\overline{X}_t}|\overline{\mathbb{P}})\mathrm{d}t + \sigma_t(X_t, \mathscr{L}_{\overline{X}_t}|\overline{\mathbb{P}})\mathrm{d}W_t$$

has strong uniqueness for some initial value X_0 with $\mathscr{L}_{X_0} = \mathscr{L}_{\overline{X}_0}|\overline{\mathbb{P}}$, then (3) has a strong solution starting at X_0. If moreover (3) has strong uniqueness for any initial value X_0 with $\mathscr{L}_{X_0} = \mathscr{L}_{\overline{X}_0}|\overline{\mathbb{P}}$, then it is weakly well-posed for the initial distribution $\mathscr{L}_{\overline{X}_0}|\overline{\mathbb{P}}$.

1. X. Huang, F.-Y. Wang, McKean-Vlasov SDEs with drifts discontinuous under Wasserstein distance., Discrete Contin. Dyn. Syst. 41(2021), no.4, 1667-1679.

Existence and Uniqueness of Solutions

Outline of Proof:

For any $\mu \in C([0,T],\mathcal{P}_2)$ and $X_0 \in L^2(\Omega \to \mathbb{H}; \mathscr{F}_0)$, the following SPDE

$$\mathrm{d}X_t = \{AX_t + b_t(X_t,\mu_t)\}\mathrm{d}t + Q_t(X_t,\mu_t)\mathrm{d}W_t \tag{4}$$

has a unique mild solution. Due to Theorem 1 and the modified Yamada-Watanabe principle, we only need to prove the strong uniqueness of DDSPDEs.

For $\nu \in C([0,T],\mathcal{P}_2)$ and $Y_0 \in L^2(\Omega \to \mathbb{H}; \mathscr{F}_0)$, Y_t solve (4) with (μ, X_0) replaced by (ν, Y_0). By the finite-dimensional approximation and Zvonkin's transform, for λ large enough it holds,

$$\int_0^l e^{-2\lambda s}\mathbb{E}|X_s - Y_s|^2\mathrm{d}s \le \frac{1}{2}\int_0^l e^{-2\lambda s}\mathbb{W}_2(\mu_s,\nu_s)^2\mathrm{d}s + c(T)\mathbb{E}|X_0 - Y_0|^2, \quad l \in [0,T].$$

For two solution $\hat{X}_t$ and $\tilde{X}_t$ with common initial value $\xi \in L^2(\Omega \to \mathbb{H}; \mathscr{F}_0)$,

$$\int_0^T e^{-2\lambda s}\mathbb{E}|\hat{X}_s - \tilde{X}_s|^2\mathrm{d}s \le \frac{1}{2}\int_0^T e^{-2\lambda s}\mathbb{W}_2(\mathscr{L}_{\hat{X}_s},\mathscr{L}_{\tilde{X}_s})^2\mathrm{d}s$$

Wang's log-Harnack Inequality and Harnack Inequality

Theorem 3 [X.Huang, S., NLA, '21]

Assume **(a1)-(a3)** and that $Q_t(x,\mu)$ does not depend on μ. Then the following assertions hold.

(1) There exists an increasing function $C : [0,\infty) \to (0,\infty)$ such that for any $T > 0$, the log-Harnack inequality

$$P_T \log f(\nu_0) \le \log P_T f(\mu_0) + \frac{C(T)}{T \wedge 1} \mathbb{W}_2(\mu_0,\nu_0)^2, \ \ \mu_0,\nu_0 \in \mathcal{P}_2$$

holds for strictly positive function $f \in \mathscr{B}_b(\mathbb{H})$. Consequently, we have

$$2\|P_T^*\mu_0 - P_T^*\nu_0\|_{\mathrm{TV}}^2 \le \mathrm{Ent}(P_T^*\mu_0|P_T^*\nu_0) \le \frac{C(T)}{T \wedge 1} \mathbb{W}_2(\mu_0,\nu_0)^2.$$

Cont.

(2) If $Q_t(x,\mu)$ does not depend on (x,μ), the Harnack inequality with power $p>1$ holds for non-negative $f\in\mathscr{B}_b(\mathbb{H})$ and any $T>0$, i.e.

$$(P_Tf(\mu_0))^p \le P_Tf^p(\nu_0)\left(\mathbb{E}\exp\left\{\frac{p}{2(p-1)^2}\Phi(T)\right\}\right)^{p-1},\quad \mu_0,\nu_0\in\mathcal{P}_2,$$

where

$$\Phi(T)=K(T)\left(4T\phi^2\left(|X_0-Y_0|\right)+C(T)\mathbb{W}_2(\mu_0,\nu_0)^2+2\frac{|X_0-Y_0|^2}{T}\right),$$

with $\mathscr{L}_{X_0}=\mu_0$ and $\mathscr{L}_{Y_0}=\nu_0$. Consequently, $P_T^*\mu_0$ is equivalent to $P_T^*\nu_0$ and it holds

$$P_T\left\{\left(\frac{\mathrm{d}P_T^*\mu_0}{\mathrm{d}P_T^*\nu_0}\right)^{\frac{1}{p-1}}\right\}(\mu_0)\le\mathbb{E}\exp\left\{\frac{p}{2(p-1)^2}\Phi(T)\right\}.$$

Proof of Log-Harnack inequality

Outline of Proof: Let $\mu_t = P_t^* \mu_0$ and $\nu_t = P_t^* \nu_0$. Let X_t be the solution to SPDEs

$$\mathrm{d}X_t = AX_t \mathrm{d}t + b_t(X_t, \mu_t)\mathrm{d}t + Q_t(X_t)\mathrm{d}W_t \tag{5}$$

with $\mathscr{L}_{X_0} = \mu_0$. Define

$$\gamma_s = Q_s^*(Q_s Q_s^*)^{-1}(X_s)[b_s(X_s, \mu_s) - b_s(X_s, \nu_s)], \qquad \bar{W}_t = W_t + \int_0^t \gamma_s \mathrm{d}s,$$

and

$$R_T = \exp\left\{-\int_0^T \langle \gamma_s, \mathrm{d}W_s \rangle - \frac{1}{2}\int_0^T |\gamma_s|^2 \mathrm{d}s\right\}.$$

By **(a2)**-**(a3)** and Girsanov's theorem, $\{\bar{W}_s\}_{s\in[0,T]}$ is a cylindrical Brownian motion under $\mathbb{Q}_T = R_T \mathbb{P}$.

Let $\bar{\mu}_t$ be the distribution of X_t under $\mathbb{Q}_T$, then $\mathbb{P}$-a.s.

$$\frac{\mathrm{d}\bar{\mu}_T}{\mathrm{d}\mu_T}(X_T) = \mathbb{E}(R_T | X_T).$$

Proof of Log-Harnack inequality

Next, consider the following equation on $(\Omega, \mathscr{F}, \mathbb{Q}_T)$

$$\mathrm{d}Y_t = AY_t\mathrm{d}t + b(Y_t, \nu_t)\mathrm{d}t + Q_t(Y_t)\mathrm{d}\bar{W}_t \tag{6}$$

If $Y_0 = X_0$, then $Y = X$. If $\mathscr{L}_{Y_0} = \nu_0$, then due to the weak uniqueness of solutions, $\mathscr{L}_{Y_t}|\mathbb{Q}_T = \nu_t$. By Log-Harnack inequality in distribution independent SPDEs, we have

$$\mathrm{Ent}(\nu_T|\overline{\mu}_T) \leq \frac{C}{T \wedge 1}\mathbb{W}_2(\mu_0, \nu_0)^2.$$

Proof of Log-Harnack inequality

By direct calculus,

$$
\begin{aligned}
P_T \log f(\nu_0) =& \mu_T \left(\frac{\mathrm{d}\bar{\mu}_T}{\mathrm{d}\mu_T} \frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \log f \right) \\
\leq & \log P_T f(\mu_0) + \mu_T \left(\frac{\mathrm{d}\bar{\mu}_T}{\mathrm{d}\mu_T} \frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \log \left(\frac{\mathrm{d}\bar{\mu}_T}{\mathrm{d}\mu_T} \frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \right) \right) \\
= & \log P_T f(\mu_0) + \bar{\mu}_T \left(\frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \log \frac{\mathrm{d}\bar{\mu}_T}{\mathrm{d}\mu_T} \right) + \bar{\mu}_T \left(\frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \log \frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \right) \\
\leq & \log P_T f(\mu_0) + \log \bar{\mu}_T \left(\frac{\mathrm{d}\bar{\mu}_T}{\mathrm{d}\mu_T} \right) + 2\bar{\mu}_T \left(\frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \log \frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \right) \\
\leq & \log P_T f(\mu_0) + \log \mathbb{E} R_T^2 + 2\bar{\mu}_T \left(\frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \log \frac{\mathrm{d}\nu_T}{\mathrm{d}\bar{\mu}_T} \right) \\
\leq & \log P_T f(\mu_0) + \frac{C(T)}{T \wedge 1} \mathbb{W}_2(\mu_0, \nu_0)^2
\end{aligned}
$$

Proof of Harnack inequality

Recall $\mu_t = P_t^*\mu_0$ and $\nu_t = P_t^*\nu_0$. Let X_t, Y_t solve the equations respectively

$$\mathrm{d}X_t = AX_t\mathrm{d}t + b_t(X_t,\mu_t)\mathrm{d}t + Q_t\mathrm{d}W_t,$$

$$\mathrm{d}Y_t = AY_t\mathrm{d}t + b_t(X_t,\mu_t)\mathrm{d}t + Q_t\mathrm{d}W_t + e^{At}\frac{X_0 - Y_0}{T}\mathrm{d}t$$

with $\mathscr{L}_{X_0} = \mu_0$ and $\mathscr{L}_{Y_0} = \nu_0$.

Theorem 4 [X.Huang, S., NLA,'21]

Assume **(a1)-(a3)**. If $Q_t(x,\mu)$ does not depend on x, then for any $T>0$, $\mu_0 \in \mathcal{P}_2$, $y \in \mathbb{H}$ and non-negative $f \in \mathscr{B}_b(\mathbb{H})$, it holds

$$(P_T f(\mu_0))^p \le P_T(f^p(e^{AT}y+\cdot))(\mu_0)\exp\left[\frac{p}{(p-1)}K(T)\Big(T\phi^2(|y|)+\frac{|y|^2}{T}\Big)\right].$$

Recall $\mu_t = P_t^*\mu_0$. Let X_t, Y_t solve the equations

$$\begin{aligned} \mathrm{d}X_t &= AX_t\mathrm{d}t + b_t(X_t,\mu_t)\mathrm{d}t + Q_t(\mu_t)\mathrm{d}W_t, \quad \mathscr{L}_{X_0}=\mu_0,\\ \mathrm{d}Y_t &= AY_t\mathrm{d}t + b_t(X_t,\mu_t)\mathrm{d}t + Q_t(\mu_t)\mathrm{d}W_t + e^{At}\frac{y}{T}\mathrm{d}t, \quad Y_0 = X_0. \end{aligned}$$

Then we have $Y_t = X_t + e^{At}\frac{ty}{T}$. In particular, $Y_T = X_T + e^{AT}y$.

Remarks

- Assume $Q_t(x,\mu) = Q_t(x)$.
  Strong well-posedness[1]: condition on the drift in the measure components can be extended to $\|\mu-\nu\|_{var,\theta} + \mathbb{W}_\theta(\mu,\nu)$ for some $\theta \geq 1$.
  Note that $\mathbb{W}_k(\mu,\nu)^{k\vee 1} \leq c(k)\|\mu-\nu\|_{var,k}, k>0$ and $\|\mu-\nu\|_{var,k_1} \leq (2-\frac{k_1}{k_2})\|\mu-\nu\|_{var,k_2}, 0<k_1\leq k_2$.
  Log-Harnack inequality[2]: $\mathbb{W}_2$ can be extended to $\mathbb{W}_k + \mathbb{W}_2$ some $k\in(0,1)$.

$$P_T \log f(\nu_0) \leq \log P_T f(\mu_0) + \frac{C(T)}{T\wedge 1}\mathbb{W}_2(\mu_0,\nu_0)^2,\ \mu_0,\nu_0 \in \mathcal{P}_2$$

1. F.-Y. Wang, Distribution dependent reflecting stochastic differential equations, arXiv:2106.12737.

2. X. Huang, F.-Y. Wang, Log-Harnack Inequality and Bismut Formula for Singular McKean-Vlasov SDEs, arXiv:2207.11536.

Remarks

- Assume $Q_t(x,\mu) = Q_t(\mu)$.
  Strong well-posedness: condition on the drift in the measure components can be extended to $\|\mu-\nu\|_{var,\theta} + \mathbb{W}_\theta(\mu,\nu)$ for some $\theta \geq 1$.
  Log-Harnack inequality[1]: b is not bounded and

  $$|b_t(x,\mu) - b_t(x,\nu)| \leq C[|x-y| + \mathbb{W}_2(\mu,\nu)], \quad K \leq QQ^* \leq K^{-1}.$$

  Then

  $$P_T \log f(\nu_0) \leq \log P_T f(\mu_0) + \frac{C(T)}{T\wedge 1}\mathbb{W}_2(\mu_0,\nu_0)^2, \ \mu_0,\nu_0 \in \mathcal{P}_2$$

  Idea: $QQ^* = K + [QQ^* - K]$. However, singular drift can be not dealt with.

1. X. Huang, F.-Y. Wang, Regularities and Exponential Ergodicity in Entropy for SDEs Driven by Distribution Dependent Noise, arXiv:2209.14619.

Remarks

- Assume $Q_t(x,\mu) = Q_t(x,\mu)$.
  In finite case, Strong well-posedness[1]: condition on drift in the measure components can be extended to $\|\mu-\nu\|_{var,\theta} + \mathbb{W}_\theta(\mu,\nu)$ for some $\theta \geq 1$.
  Log-Harnack inequality: Open problem, even in finite case.

1. X. Huang, F.-Y. Wang, Singular McKean-Vlasov (reflecting) SDEs with distribution dependent noise. J. Math. Anal. Appl. 514 (2022), no. 1, Paper No. 126301, 21 pp.

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Thank You for Your Attention !